Four-arm polyominoes in Golomb's hierarchy

A complete classification with Lean verification

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Abstract

We classify the polyominoes obtained by adjoining four straight arms to a single square, allowing zero arm lengths, according to their ability to tile rectangles, half-strips, bent strips, quadrants, strips, half-planes, and the plane. We also classify their ability to tile an integer enlargement of themselves. Tiles occupy whole square-grid cells; translations, rotations, and reflections are permitted. Exactly five capability profiles occur. For the family $P(n, 1, 1, 0)$, the rectangle profile holds for $n \leq 3$ and the bent-strip profile, with no half-strip or rep-tiling, for every $n \geq 4$. A cross with four positive arms tiles the plane precisely when two opposite arms have length one; it never tiles a half-plane. Explicit periodic constructions and geometric obstructions are combined with finite symbolic case certificates. A Lean 4 development verifies the full classification for every natural four-tuple, including the interpretation of the certificates as statements about arbitrary infinite tilings. The account incorporates the author's 2020–2021 L- and T-polyomino work, reconstructs Dahlke's gun argument, and documents the subsequent AI-assisted proof development and formalization.

Contents

1 Introduction and provenance

Golomb's hierarchy distinguishes several increasingly permissive regions that a single polyomino may tile [5]. A plane tiling need not accommodate a straight boundary, and a strip tiling need not admit a cap. Determining these distinctions for an infinite parametric family requires both constructions and obstructions valid without a bound on the parameters or on the region widths. General questions about the hierarchy remain a subject of study [12]. Rectangle and half-strip tileability have also been studied systematically by Reid [9].

This paper concerns the union of a horizontal and a vertical row of cells meeting at a distinguished cell. Zero-length arms include the bar, L, and T families. We give a single explicit decision rule for all eight tiling capabilities considered here and prove that rule in Lean. The mathematical results and their precise grid convention are stated in Section 2. Some proofs are short geometric arguments; others use exhaustive finite case analysis with symbolic integer parameters. The soundness principle for the latter is stated in Theorem 3.2. An interactive companion article gives a tuple classifier and additional illustrations; the paper page collects the manuscript, sources, citation, and formal artifacts.

1.1 Earlier work and the present development

The author's original research was carried out between October 2020 and March 2021, while at school. The L results were presented in two 2021 conference papers [7, 8]. The author also obtained the T classification during that period, apart from $P(4, 1, 1, 0)$ and $P(5, 1, 1, 0)$, in handwritten work that was not prepared as a digital publication. Those handwritten T proofs were not provided to the AI assistant used for the present development. The reconstructed T results agree with the author's earlier classification. The dates of the research are distinct from the later presentation dates.

Dahlke's *Gun Theorem* is a direct source for the argument concerning $P(n, 1, 1, 0)$ [1]. Its section on a one-cell face treats every $n \geq 4$ and uses three supported-corner configurations to rule out rectangles. Our formal corner system makes the reflected configurations explicit and proves additional progress and containment inequalities for half-strips and rep-tilings. We do not attribute the underlying corner-propagation method, or rectangle nonrectifiability of the two small cases, to the present investigation. The theorem was documented in the revision of Tulleken's book dated 19 May 2019 [11]; the original composition date of Dahlke's online argument is unspecified. The half-strip and rep-tile conclusions are established here with explicit certificates and formal proofs. We make no claim that these conclusions could not also be extracted from the earlier argument.

The rectangle tiling of the Y hexomino used below is a historical construction, attributed to Marlow and Dahlke in Reid's record [2, 6, 10]. It was converted to cell coordinates and independently verified; its discovery and minimality are not contributions of this paper. The L classification is likewise credited to the original papers rather than to its formal reconstruction. Dahlke's separate *L Theorem*, completed in its repository in September 2020, studies rectangle tileability for a broader family of thick L shapes [3]. That related result is distinct in scope from the half-plane classification used here.

The present contribution is a unified, explicit classification with constructions, complete obstruction certificates, and a formal proof of every capability for every tuple. It also documents the genuine-cross analysis developed during the subsequent investigation. This contribution statement does not assert priority for every individual tiling or local lemma, and does not identify the number of generated proof obligations with the intrinsic difficulty of the underlying mathematics.

1.2 Use of generative AI

OpenAI's Astra 6 model, operating through the Codex harness, was used extensively in the development of this paper and its accompanying artifacts. Astra 6's role included mathematical exploration, reconstruction and development of proofs, discovery of constructions, counterexample searches, certificate generation, Lean formalization, drafting, and preparation of figures and source packages. In particular, the work completing the two exceptional T cases, the genuine-cross analysis, and the formal development were produced through this AI-assisted investigation initiated and directed by the author. This description distinguishes those contributions from the author's earlier work and from the cited literature. AI assistance was substantive and was not confined to language editing. Lean verification and independent certificate checks concern the stated mathematical assertions; they do not themselves establish historical priority or the quality of the exposition. The author is responsible for the submitted manuscript.

2 Definitions and the complete classification

Throughout, $\mathbb{N} = \{0, 1, 2, \ldots\}$. Identify the cell $[x, x+1] \times [y, y+1]$ with $(x, y) \in \mathbb{Z}^2$; disjointness below means disjoint cell sets, so neighboring closed squares may share edges. For $a, b, c, d \in \mathbb{N}$ define

$$P(a, b, c, d) = \{(x, 0) : -c \leq x \leq a\} \cup \{(0, y) : -d \leq y \leq b\}. \tag{1}$$

The arm parameters run east, north, west, and south and exclude the junction. Thus the area is $a + b + c + d + 1$. A copy is an integer translate of any of the eight square-grid rotations or reflections. A tiling of a region is an arbitrary collection of copies contained in the region, pairwise disjoint, whose union is the region. No periodicity assumption is imposed on an infinite tiling.

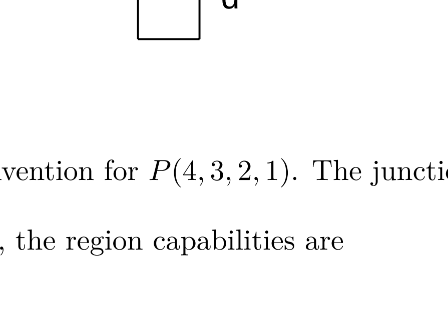

Figure 1: The convention for $P(4, 3, 2, 1)$. The junction is counted once.

For positive integers u, v, w, the region capabilities are

Symbol	Some region of the following form is tileable
R	$\{0,\dots,u-1\}\times\{0,\dots,v-1\}$
HS	$\mathbb{N}\times\{0,\dots,w-1\}$
BS	$\{(x,y)\in\mathbb{N}^2 : x<u \text{ or } y<v\}$
Q	$\mathbb{N}^2$
S	$\mathbb{Z}\times\{0,\dots,w-1\}$
HP	$\mathbb{Z}\times\mathbb{N}$
Plane	$\mathbb{Z}^2$

Every width quantifier is existential and ranges over all positive integers. A failure of HS, for example, excludes every such width. The capability Rep means that copies tile an enlargement of the polyomino by a natural integer linear factor $k\ge 2$: every original cell is replaced by its $k\times k$ block. This is the lattice rep-tile convention. Off-grid placements and dissections into unequal-scale copies are outside the statement.

The region implications are

$$\mathrm{R}\implies\mathrm{HS}\implies\mathrm{BS}\implies\begin{cases}\mathrm{Q}\\ \mathrm{S}\end{cases}\implies\mathrm{HP}\implies\mathrm{Plane},\qquad \mathrm{R}\implies\mathrm{Rep}\implies\mathrm{Q}. \tag{2}$$

They do not form a single chain. Define the following *profiles*, listing all positive capabilities and excluding every unlisted one:

Profile	R	Rep	HS	BS	Q	S	HP	Plane
Rectangle	yes	yes	yes	yes	yes	yes	yes	yes
Bent strip	–	–	–	yes	yes	yes	yes	yes
Strip	–	–	–	–	–	yes	yes	yes
Plane only	–	–	–	–	–	–	–	yes
Non-tiler	–	–	–	–	–	–	–	–

Theorem 2.1 (Complete classification)**.** *Every polyomino in* (1) *has exactly the profile given below. For an L shape use $P(a,b,0,0)$ with $a\ge b\ge 1$; for a T shape use $P(a,b,c,0)$ with $a\ge c\ge 1$ and $b\ge 1$. These normal forms are obtained by square-grid symmetries. Bars include the single cell and all representations with only two opposite positive arms.*

Normalized shape and parameter condition	Profile
Bar	Rectangle
L: $b=1$	Rectangle
L: $b=2$ or $(a,b)=(4,3)$	Strip
L: all other $a\ge b\ge 1$	Plane only
T: $b=c=1$, $a\le 3$	Rectangle
T: $b=c=1$, $a\ge 4$	Bent strip
T: $b=1$, $c\ge 2$	Strip
T: $b\ge 2$ and at least one of $c=1$, $b=2$, or $(c=2,\ b=a+3)$	Plane only
T: all remaining parameters	Non-tiler
Cross: $a,b,c,d\ge 1$, with $a=c=1$ or $b=d=1$	Plane only
Cross: all remaining positive parameters	Non-tiler

Corollary 2.2. *For every $n \in \mathbb{N}$, $P(n, 1, 1, 0)$ has the rectangle profile if $n \leq 3$ and the bent-strip profile if $n \geq 4$.*

Corollary 2.3. *A genuine T in normal form tiles the plane if and only if $b \leq 2$, or $c = 1$, or ($c = 2$ and $b = a + 3$). A genuine cross tiles the plane if and only if two opposite arms have length one. In particular, four arms all greater than one give a non-tiler.*

The five profiles are exhaustive for this family, not for arbitrary polyominoes. The proofs below establish the positive constructions and the separating obstructions; (2) then determines each remaining capability. The normalization of every natural tuple, including duplicate descriptions of bars, is also part of the formal theorem.

3 Hierarchy implications and finite certificates

Proposition 3.1. *All implications in (2) hold for the lattice convention of Section 2.*

Proof. Repeated rectangles give a half-strip. A half-strip of width w and a rotated copy beginning at $(0, w)$ give a bent strip with widths w, w. For a bent strip of widths u, v, its translates by $k(u, v)$, $k \in \mathbb{N}$, partition the quadrant: the unique layer containing (x, y) has index $\min(\lfloor x/u \rfloor, \lfloor y/v \rfloor)$. A full strip gives a half-plane by parallel repetition.

For the remaining infinite-region implications use finite local choice. Only finitely many copies of a fixed finite tile contain a specified cell. Translate a sequence of tiled regions so that increasingly large finite patches of the target region are covered and its required boundary is respected. Successively restrict to subsequences that agree on every larger patch. The consistent limit consists of whole copies, remains disjoint, and covers every target cell. Moving far down one arm of a bent strip gives a strip; moving far along the boundary of a quadrant gives a half-plane; moving into the interior of a half-plane gives the plane. This argument asserts existence of a limiting tiling and does not assert periodicity.

If an $r \times s$ rectangle is tileable, choose an integer $k > 1$ divisible by both r and s. Every $k \times k$ block is tiled by such rectangles, so replacing each cell of the prototile by a block proves Rep. Conversely, iterate a rep-tiling and place a fixed convex corner of each successively enlarged tile at the origin. For any prescribed bounded quadrant patch, sufficiently large enlargements contain the patch and have the two required adjacent boundary segments. Since tile diameters are bounded, a copy meeting the prescribed patch cannot reach past either of those increasingly long boundary segments. The same finite-choice argument gives a quadrant tiling. $\square$

3.1 What a negative certificate proves

A finite search box is not itself an obstruction to an infinite tiling: a tile may extend beyond the box. Our certificates instead query cells of a hypothetical actual tiling and retain every cell of each selected copy. The arithmetic parameters may be fixed or range over an explicitly specified unbounded domain.

Theorem 3.2 (Soundness of finite obstruction trees)**.** *Fix a parameter domain, a region, and a finite collection of whole selected tiles required to occur in a hypothetical exact tiling. Suppose there is a finite rooted tree with the following properties for every parameter choice in the domain. Each nonterminal node specifies a cell of the region not covered by its selected tiles. Its children exhaust every congruent whole tile through that cell which is contained in the region and disjoint from the* *selected tiles. Each child adds that tile. Every terminal node has a proved contradiction, either no possible cover or a local obstruction whose hypotheses hold for the selected tiles. Then no exact tiling contains the root configuration.*

Proof. Assume such a tiling exists. At a nonterminal node its queried cell has a unique covering tile in the tiling. That tile satisfies the containment and disjointness conditions, so exhaustiveness gives a child whose selected tiles also belong to the tiling. Induction on the finite tree, starting at its leaves, contradicts the terminal obstruction. For symbolic parameters fix an arbitrary admissible tuple first; the same induction then applies because each branch implication was proved throughout the domain. □

An analogous certificate may terminate in a specified successor configuration instead of a contradiction. The same induction proves the local transition statement. Section 5 combines such transitions with a well-founded rank. This distinction is necessary: a finite transition certificate is not by itself an infinite-region obstruction.

The certificates enumerate all eight orientations and allow an arbitrary integer junction for the covering tile. A proposed child can be impossible for some parameter values; that branch is then vacuous, not an omitted case. Search and external solvers propose the trees and arithmetic claims. The formal development checks the arithmetic and proves its connection to copies, containment, coverage, and disjointness in an actual tiling. The finite symbolic trees used later are explicit parts of the proof, available in the versioned artifacts. A named computer-assisted proposition below therefore refers to a supplied exhaustive proof, not to unsuccessful testing over finitely many parameter values.

4 Explicit positive constructions

A lattice construction below specifies finitely many copies and a rank-two translation lattice. It proves a plane tiling when the cells of the copies represent each class of $\mathbb{Z}^2/\Lambda$ exactly once. This verifies both coverage and nonoverlap, not just agreement of areas. Write $\sigma(x, y) = (y, x)$ and $-T = \{(-x, -y) : (x, y) \in T\}$.

4.1 Bars and small rectangle cases

A bar is already a rectangle. For $P(1, 1, 1, 0)$ and $P(2, 1, 1, 0)$ the following partitions give rectangles; each letter denotes one complete copy. Checking the cells of each letter gives the required congruences.

$P(1,1,1,0)$	$P(2,1,1,0)$

`BCCC`
`BBCD`
`BADD`
`AAAD`
	`BBBBEHHHHJ`
	`ABEEEEHGJJ`
	`ADDDDGGGGJ`
	`AACDFFFFIJ`
	`ACCCCFIIII`

The historical 24×23 rectangle of 92 Y hexominoes provides the $P(3, 1, 1, 0)$ case [2, 10]; see Figure 2. The formal artifact checks all 552 cells exactly once against the explicit 92-copy witness. Only existence is needed for this classification.

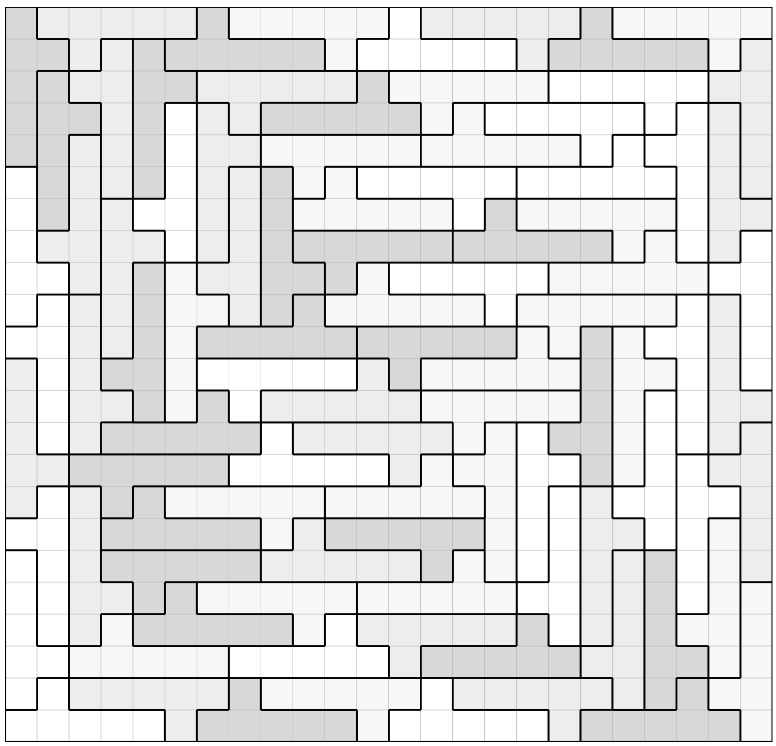

Figure 2: A cell-coordinate reconstruction of the Y-hexomino rectangle recorded by Reid and attributed to Marlow and Dahlke. Each region is one copy of $P(3,1,1,0)$.

4.2 Strips and bent strips for T shapes

Proposition 4.1. *Every $T = P(a, 1, c, 0)$ with $a, c \geq 1$ tiles a strip of width two. If $c = 1$, it also tiles a bent strip of widths two and two.*

Proof. Put $L = a + c + 1$ and $m = L + 1$. For each $j \in \mathbb{Z}$ place

$$T + (jm + c, 0), \qquad -T + (jm + L, 1).$$

On row zero the first tile covers $jm, \ldots, jm + L - 1$ and the second the cell $jm + L$. On row one the first contributes $jm + c$ and the second $jm + c + 1, \ldots, jm + c + L$. Each row is a disjoint partition into successive blocks of length m; every tile remains in the two rows.

For $c = 1$ restrict to $j \geq 0$. The tiled region is

$$J = \{(x, 0) : x \geq 0\} \cup \{(x, 1) : x \geq 1\}.$$

Its reflected translate $\sigma J + (0, 1)$ is $\{(0, y) : y \geq 1\} \cup \{(1, y) : y \geq 2\}$. These regions are disjoint and their union is $\{(x, y) \in \mathbb{N}^2 : x < 2 \text{ or } y < 2\}$. □

4.3 Three plane constructions for T shapes

Proposition 4.2. *A genuine T in normal form tiles the plane whenever $c = 1$, $b = 2$, or ($c = 2,\ b = a + 3$).*

Proof. A unit crossbar arm. Set $T = P(a, b, 1, 0)$, $m = a + b + 2$, and $\Lambda = \langle (2m, 0), (2, 1) \rangle$. Repeat the two copies T and $-T + (2a + 1, 0)$ over Λ. The quotient coordinate is $x - 2y \pmod{2m}$. The two horizontal bars supply $-1, \ldots, 2a + 2$. The first stem supplies the remaining even residues $2a + 4, 2a + 6, \ldots, 2m - 2$; the second supplies the remaining odd residues $2a + 3, 2a + 5, \ldots, 2m - 3$. These sets partition all $2m$ residues.

A two-cell stem. Set $T = P(a, 2, c, 0)$, $m = a + c + 3$, and $\Lambda = \langle (m, 0), (c + 1, 2) \rangle$. Repeat T and $-T + (a + 1, 1)$. Use quotient coordinates

$$j = y \bmod 2, \qquad u = x - \lfloor y/2 \rfloor (c + 1) \pmod{m}.$$

For $j = 0$ the first bar supplies $[-c, a]$, its upper stem endpoint supplies $-c-1$, and the second stem supplies $a + 1$. These are m consecutive residues. For $j = 1$ the second bar supplies $1, \ldots, m - 2$; the two remaining stem cells supply $0, m - 1$.

The four-copy family. Set $b = a + 3 \geq 5$ and $T = P(a, b, 2, 0)$. Repeat the four copies

$$T, \quad -\sigma T + (b - 2, -2), \quad -T + (b - 1, 1), \quad \sigma T + (1, 3) \tag{3}$$

over $\Lambda = \langle (4, 4), (b, -b) \rangle$. Each tile has $2b$ cells, and the lattice index is $8b$. Put $d = x - y$, $q = \lfloor d/(2b) \rfloor$, $\delta = d - 2bq$, and $\eta = y + bq \pmod 4$. These are coordinates on the quotient. Dividing each tile into its bar (including the junction) and its stem (excluding the junction) gives the following four representatives at each δ; the entries are y coordinates after reducing d to δ, before reduction modulo four.

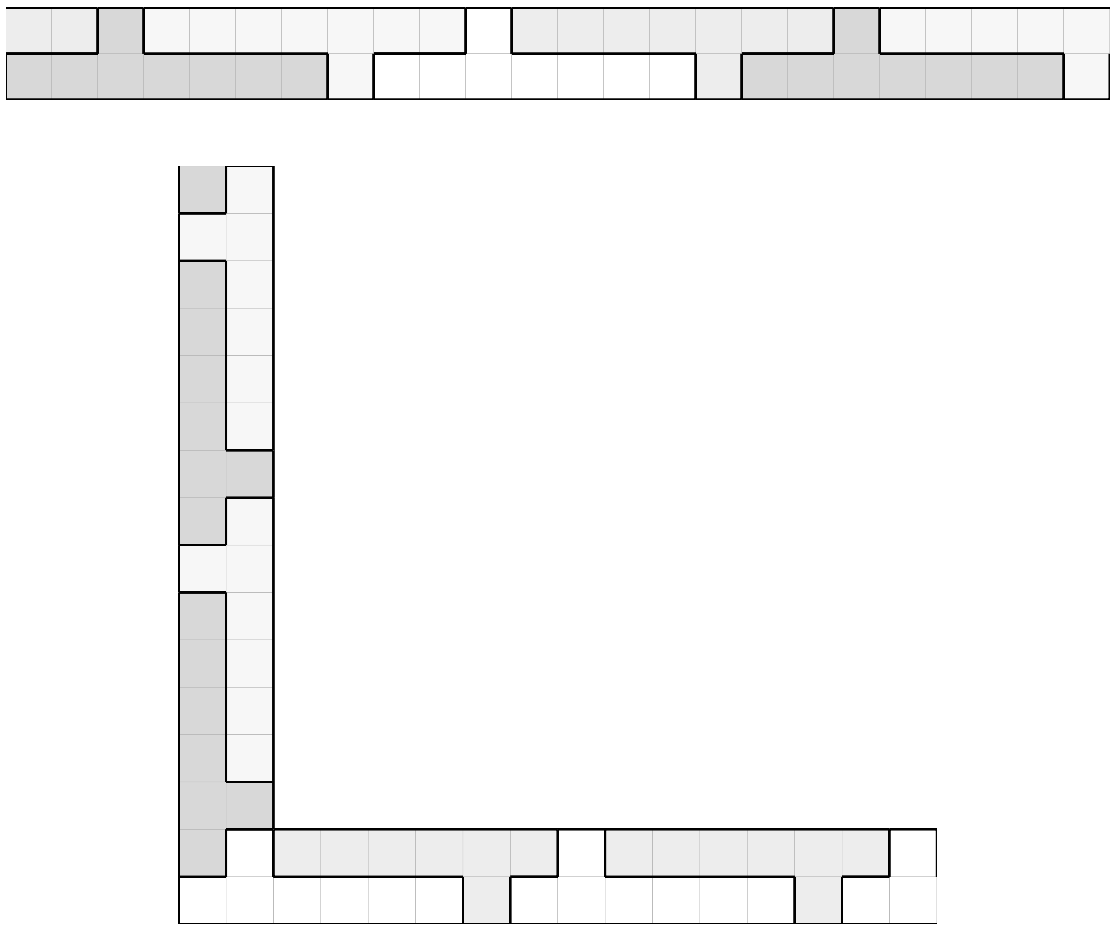

Figure 3: A strip for $P(4,1,2,0)$ (top) and a bent strip for $P(4,1,1,0)$ (bottom). The drawings are cropped; the cut ends do not assert finite rectangle tilings.

δ	Four representatives
0	$0, -2, 3, 1$
$1 \le \delta \le b-3$	$0, -2, 1, 3$
$b-2$	$-2, 1, 3, 0$
$b-1$	$-2, 1, -1, 0$
b	$1, 0, -2, -1$
$b+1 \le \delta \le 2b-3$	$b-\delta, b-2-\delta, b-1-\delta, b+1-\delta$
$2b-2$	$-b, 2-b, 1-b, 3-b$
$2b-1$	$-b, 3-b, 1-b, 2-b$

Every row is a complete set modulo four; the listed intervals exhaust $0 \le \delta < 2b$. This verifies (3) for all $a \ge 2$. $\square$

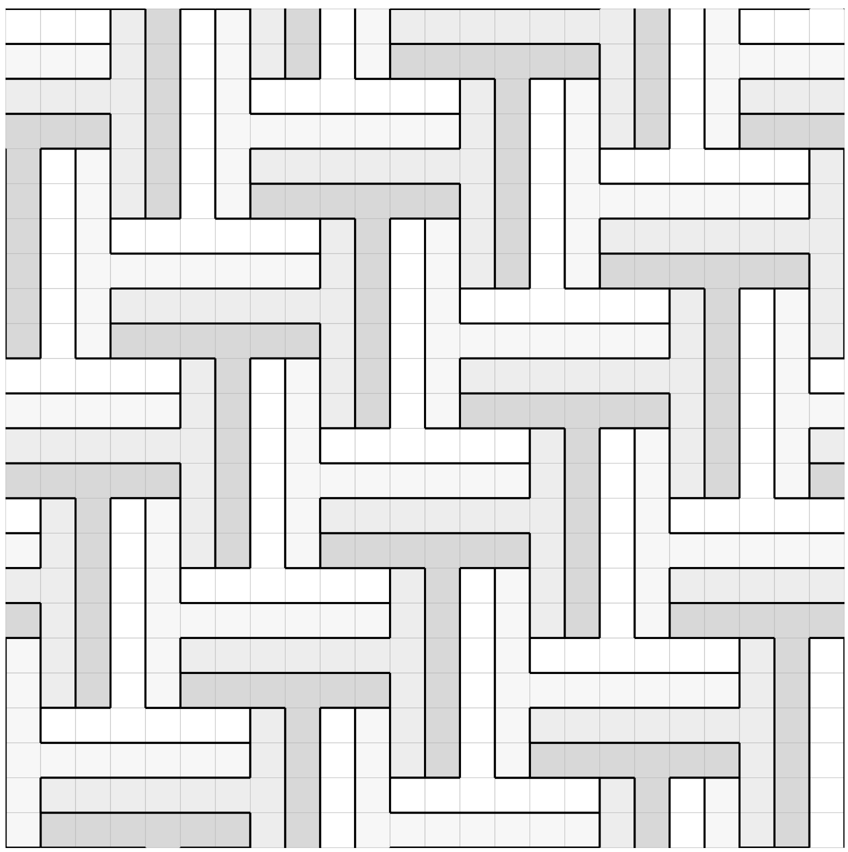

Figure 4: The four-copy plane construction for $P(3, 6, 2, 0)$. Boundary pieces are cropped by the drawing window.

4.4 Crosses with opposite unit arms

Proposition 4.3. *For $a, c \ge 1$, $C = P(a, 1, c, 1)$ tiles the plane.*

Proof. Put $m = a + c + 3$ and repeat C and $-C + (a + 2, 1)$ over $\Lambda = \langle (m, 0), (1, 2) \rangle$. Use quotient coordinates $j = y \bmod 2$ and $u = x - (y-j)/2 \pmod{m}$. For even parity the first bar supplies $-c, \ldots, a$, and the second tile's tips supply $a+1, a+2$. For odd parity the second bar supplies $2, \ldots, m-1$, and the first tile's tips supply $0, 1$. Both parity classes are partitioned exactly once. $\square$

4.5 L constructions

The positive L families are those in the author's half-plane classification [7]. Here are explicit witnesses in the arm convention.

For $L = P(a,1,0,0)$, the copies L and $-L + (a+1, 1)$ partition an $(a+2) \times 2$ rectangle. Every $L = P(a,b,0,0)$ tiles the plane: under $x - y \pmod{a+b+1}$ its cells give the consecutive residues $-b, \ldots, a$. Thus translates over $\langle (a+b+1, 0), (1,1) \rangle$ partition the grid.

For $L = P(a,2,0,0)$, $a \geq 2$, the following six tiles, repeated by $(6j, 0)$ for $j \in \mathbb{Z}$, tile the strip of height $a+3$. Each listed bar and foot includes the junction, which is counted once.

Tile	Vertical bar	Horizontal foot
A	$x = 0,\ 0 \leq y \leq a$	$y = 0,\ 0 \leq x \leq 2$
B	$x = 1,\ 1 \leq y \leq a+1$	$y = a+1,\ -1 \leq x \leq 1$
C	$x = 2,\ 2 \leq y \leq a+2$	$y = a+2,\ 0 \leq x \leq 2$
D	$x = 3,\ 2 \leq y \leq a+2$	$y = a+2,\ 3 \leq x \leq 5$
E	$x = 4,\ 1 \leq y \leq a+1$	$y = 1,\ 2 \leq x \leq 4$
F	$x = 5,\ 0 \leq y \leq a$	$y = 0,\ 3 \leq x \leq 5$

Rows $2, \ldots, a$ receive one cell from each vertical bar. Row zero is covered by the feet of A and F. Row one receives A, B, the foot of E, and F, covering residues $0, 1, 2, 3, 4, 5$. Row $a+1$ receives the foot of B at residues $5, 0, 1$ and C,D,E at $2, 3, 4$; row $a+2$ receives the feet of C and D. This proves exact coverage in every row.

For the remaining case $P(4,3,0,0)$, the following placements give a strip of height eight and horizontal period eight. A placement is written as its junction and its directed arms.

Junction	(E, N, W, S)
$(4,1)$	$(0,3,4,0)$
$(3,2)$	$(0,4,3,0)$
$(5,0)$	$(0,4,3,0)$
$(2,7)$	$(3,0,0,4)$
$(7,5)$	$(0,0,3,4)$
$(8,6)$	$(0,0,4,3)$
$(6,0)$	$(3,4,0,0)$
$(9,7)$	$(0,0,3,4)$

Reduction modulo eight in the horizontal coordinate gives each of the 64 strip cells exactly once. The motif is also checked in Lean. The drawing in the original half-plane paper can be reconstructed as a width-eight construction although its text calls the width six. This dimensional correction leaves the existence classification unchanged; it does not assert impossibility of every width-six strip.

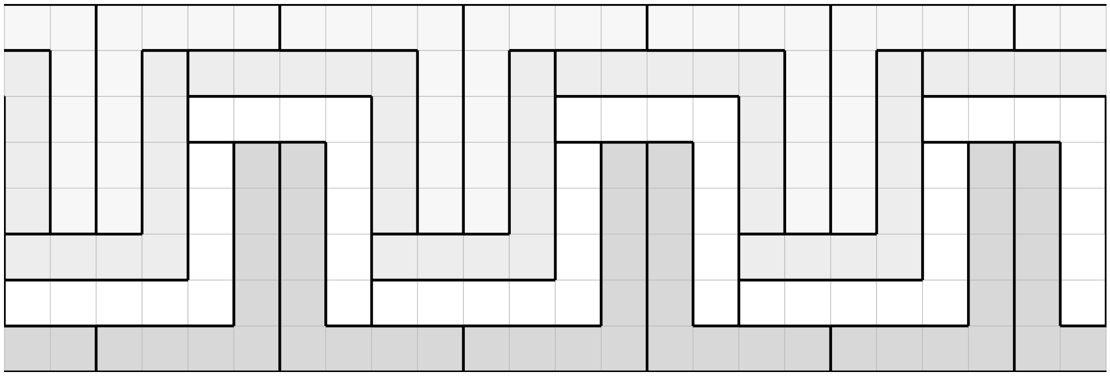

Figure 5: Three periods of the width-eight strip for $P(4,3,0,0)$, whose bounding dimensions are five by four.

5 The one-cell-stem family and Dahlke's corner argument

The positive bent-strip construction for every $P(a,1,1,0)$ was given in Proposition 4.1. This section proves the half-strip and rep-tile obstructions for every $a \geq 4$. The three basic corner configurations come directly from section 23 of Dahlke's gun argument [1]. We reconstruct its local cases as complete certificates and add explicit inequalities suitable for the two infinite or scaled-region deductions below. The rectangle obstruction is therefore credited to the earlier argument, including $a = 4, 5$.

5.1 A local transition statement

A *supported corner* at (X, Y) consists of a selected collection of whole tiles and occupied boundary cells. Relative to (X, Y) its wall contains $(-1, 0), \ldots, (-1, w-1)$ and its floor contains $(0, -1), \ldots, (f-1, -1)$. In the northeast quadrant the selected cells are exactly the decoration E specified below. Cells of a selected tile outside this quadrant remain part of that tile; they are not cut away. Initially the wall and floor can be supplied by the exterior of the target region. Interior successors use actual tile cells.

State s	w	f	Decoration E	Rank $r(s)$	Potential $h(s)$
α	4	4	$\varnothing$	0	0
β	5	6	$\{(0,0)\}$	1	2
γ	5	6	$\{(0,0),(1,0)\}$	2	2
β^{T}	6	5	$\{(0,0)\}$	1	1
γ^{T}	6	5	$\{(0,0),(0,1)\}$	0	0

The superscript denotes reflection in $x = y$.

Proposition 5.1 (Certified corner transition)**.** *For $a \geq 4$, a supported corner of type s in an exact tiling by $P(a,1,1,0)$ has a supported successor of type s' obtained by adding finitely many whole tiles, at displacement $(\Delta x, \Delta y)$, provided the supported queries lie in the target region and the covering tiles remain below its specified ceiling. The selected and supporting cells must lie below that ceiling; whenever a selected cell belongs to a tile, that whole tile is selected. The certificates*

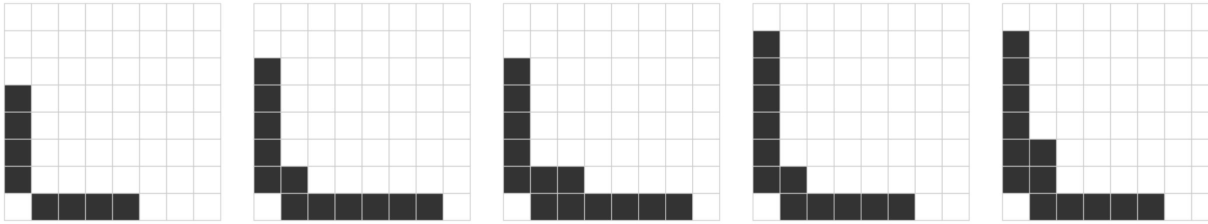

Figure 6: The five supported-corner types. Dark cells are occupied. The two reflected types make the orientation bookkeeping explicit.

query only relative columns $0, \ldots, 7$ *and prove*

$$\Delta x, \Delta y \geq 0, \qquad \Delta x + \Delta y > 0, \tag{4}$$
$$s = \alpha \implies \Delta x > 0 \text{ and } \Delta y > 0,$$
$$\Delta y = 0 \implies r(s') < r(s), \tag{5}$$
$$\Delta x - 4\Delta y \leq h(s) - h(s'). \tag{6}$$

The new selected cells meet the successor quadrant in exactly its listed decoration. In particular, they supply the successor wall and floor.

Computer-assisted proof. The local statement is supplied separately by finite certificates for $a = 4$ and $a = 5$ and by one symbolic family for all $a \geq 6$. At each node all whole copies through the queried cell are exhausted; overlaps are rejected and all surviving terminal nodes give a specified successor. Finite induction as in Theorem 3.2 proves the transition. The certificates for four and five contain 1750 and 872 nodes respectively; the unbounded family is proved using 8309 arithmetic lemmas and five geometric transition replays. The zero-height transitions can only follow $\gamma \to \beta^{\mathsf{T}} \to \alpha$ or $\beta \to \alpha$; some are absent for $a = 4$. All four inequalities and the validity of each supported query are checked in the formal replay. The formal inputs and exact whole-tiling interpretation are given in Section 11. More precisely, a query's height is bounded by that of an already occupied support cell, or by the top of an already selected whole tile. These bounds are part of the replay. If the known supports lie below a ceiling, the query therefore lies below it *before* its covering tile is requested. Containment of that newly selected tile preserves the same invariant for subsequent queries. This is the supported-query condition used in both applications below. □

These conditions state more than the existence of an endless sequence of corners. For a rectangle, unbounded $X + Y$ would suffice for a contradiction. In a half-strip, horizontal escape must also be excluded; (5) supplies that additional information. For an enlarged T, (6) controls the distance from the stem.

Theorem 5.2. *For every* $a \geq 4$, $P(a, 1, 1, 0)$ *tiles no half-strip and no rectangle.*

Proof. If a half-strip of positive height H were tileable, stack four copies to obtain one of height $K = 4H \geq 4$. At its initial corner the exterior boundary supplies state α. At any subsequent corner the queried cells are in the half-strip and whole tiles stay below its ceiling; Proposition 5.1 therefore applies. For a corner of type s at height Y assign the integer

$$3(K - Y) + r(s).$$

It is nonnegative because the supported corner lies below the ceiling. If a transition has $\Delta y = 0$, (5) decreases the rank. If $\Delta y \geq 1$, the first term decreases by at least three while r can increase by at most two. Each transition therefore strictly decreases a nonnegative integer, a contradiction. A rectangle would give a half-strip by repetition. $\square$

Theorem 5.3. *For every $a \geq 4$, $P(a, 1, 1, 0)$ is not a lattice rep-tile.*

Proof. Iterating a hypothetical rep-tiling provides integer scales as large as needed; choose $k \geq a+2$. Reflect the enlarged T and measure coordinates leftward and upward from the right-hand end of its long arm. Its cell set becomes

$$\{0 \leq X < (a+2)k,\ 0 \leq Y < k\} \ \cup\ \{ak \leq X < (a+1)k,\ k \leq Y < 2k\}.$$

Before $X = ak$ the region has height k. Start the α certificate at $(0, 0)$. A tile through a query extends at most $a+1$ cells horizontally from that query, since the largest difference of two horizontal cell coordinates in any copy is $a+1$. Thus the initial queried tiles have $X \leq a+8 < ak$ and remain in the rectangular arm. They consequently stay below height k, so the local transition hypotheses hold. Its first successor has both coordinates positive.

Summing (6) gives the invariant $X - 4Y \leq -h(s)$. For an interior corner still before the stem, its actual vertical wall of height $w \geq 4$ lies in that arm, so $Y + w \leq k$. Consequently

$$ak - X \geq (a-4)k + 4w + h(s) > a + 8. \tag{7}$$

For the strict inequality, subtract $a + 8$ from the middle expression:

$$(a-4)(k-1) + 4(w-4) + h(s) + 4 \geq 4.$$

Every whole tile through the next query therefore remains before the stem, since the query is at most seven columns to the right of the corner. Coverage and containment validate all supported queries below the height-k ceiling. The successor floor is supplied by these selected cells, so its origin also remains before the stem. This proves the invariant and clearance inductively, without extending the rectangular arm across the concave boundary. The same rank $3(k - Y) + r(s)$ as in Theorem 5.2 now strictly decreases indefinitely, a contradiction. $\square$

Together with the rectangle witnesses for $a \leq 3$, the bent-strip construction, and Proposition 3.1, these two theorems prove Corollary 2.2, including the rep-tile entry. The case $n = 0$ in that corollary is an L triomino and has a two-copy rectangle construction from Section 4.5.

6 Boundary obstructions for T-polyominoes

Throughout this section let $T = P(a, b, c, 0)$, where $a \geq c \geq 1$ and $b \geq 1$, and put $A = a + c + 1$. The *crossbar* has A cells; the perpendicular *stem* has b cells beyond its junction. All coordinates below denote cells, rather than vertices.

Lemma 6.1 (The quadrant corner)**.** *If $c \geq 2$, then T does not tile a quadrant.*

Proof. A tile covering $(0,0)$ in $\mathbb{N}^2$ has a crossbar endpoint there: a stem contact would put one crossbar arm outside the quadrant. After diagonal reflection, its crossbar is $[0, A-1] \times \{0\}$ and its upward stem has abscissa $r \in \{a, c\}$.

The tile covering $(0,1)$ cannot have a horizontal crossbar, since it would meet that stem. A horizontal stem contact would put a vertical crossbar arm below row zero, since both such arms have length at least two. Thus its vertical crossbar must start at $(0,1)$ and its stem point right. Now $(1,1)$ has no cover. A horizontal crossbar through it must begin at $x = 1$ and intersects the first stem; a vertical crossbar must begin at $y = 1$ and intersects the second stem. Either perpendicular-stem contact puts a crossbar arm beyond the quadrant. □

We use the following small-gap argument repeatedly. The occupied supports in its statement are unions of *whole tiles* selected from a hypothetical tiling. A tile meeting an unoccupied cell is therefore disjoint from all these supports.

Lemma 6.2 (Blocked short run)**.** *Suppose $b \leq A-1$. A run of g consecutive cells in one row, where $3 \leq g < A$, cannot be filled if its immediately lower neighbors and its two immediate horizontal neighbors are occupied, while the run is disjoint from the supporting tiles.*

Proof. A horizontal crossbar cannot fit. A horizontal stem contact either crosses an occupied endpoint or puts its junction above the occupied floor, where a downward crossbar arm collides with the floor. Each cell consequently starts either a vertical crossbar, called V, or an upward vertical stem whose junction is above it, called S. Their vertical segments have respectively A and $b+1$ cells. An S crossbar intersects its neighboring starter at height b above the run. Thus all starters must be V. Two adjacent V tiles can be disjoint only when their stems point away from one another; three consecutive V tiles are impossible. □

The same last argument shows that a consecutive block of V/S starters has length at most two when $b \leq A-1$, even if horizontal crossbars occupy other parts of the row.

Theorem 6.3. *If $b \geq 2$, then T does not tile a half-plane.*

Proof. Work in $y \geq 0$. A boundary tile has one of three forms: H, with its horizontal crossbar on row zero and stem up; V, with its vertical crossbar starting on row zero; or S, with the tip of its vertical stem on row zero and horizontal crossbar at height b.

First, if $b > a$, a rightward boundary V at $x = 0$, with junction height $r \in \{a, c\}$, is impossible. At $(1,0)$ a V or S intersects its stem at $(1,r)$. An H must begin at $x = 1$ and has stem at $1+s$, $s \in \{a, c\}$; this intersects the old stem because $1 + s \leq a + 1 \leq b$ and $r < b$. Reflect for a leftward V.

Next exclude S for every $b \geq 2$. Put its tip at $(0,0)$ and its length-a crossbar arm to the right. Its neighbor $(1,0)$ cannot be S. It cannot be V: for $b \leq A-1$ there is an intersection at $(1,b)$, and otherwise the preceding exclusion applies. Hence this neighbor begins an H. Avoiding the old crossbar forces its junction to $x = a+1$. The two tiles enclose the nonempty rectangle

$$\{1, \dots, a\} \times \{1, \dots, b-1\}.$$

Any tile meeting its interior must lie wholly inside, by connectivity and the occupied boundary. A horizontal crossbar is too wide; a vertical crossbar would require both $b+1 \leq a$ and $A \leq b-1$, which are incompatible.

Assume first $c \geq 2$. For $b \geq A-1$, V has already been excluded. Otherwise put a proposed rightward V at $x = 0$ and its junction at height $r \in \{a, c\}$. The cell $(1, 0)$ must begin an H whose stem is at $j = 1 + s$, $s \in \{a, c\}$. The unoccupied cell $(1, 1)$ has no cover: a horizontal crossbar hits the H stem; a vertical crossbar hits the V stem; a vertical stem contact either overlaps row zero or puts a horizontal arm across $x = 0$ below height $b + 2 \leq A$; a horizontal stem contact puts a crossbar arm below row zero. Thus only H tiles meet the boundary.

Their crossbars partition row zero. If consecutive junction offsets are $r_k, r_{k+1} \in \{a, c\}$, the gap between their stems in row one has width

$$g_k = A - 1 + r_{k+1} - r_k \geq 2c. \tag{8}$$

Some consecutive pair satisfies $r_{k+1} \leq r_k$, since an infinite sequence in a two-element set cannot strictly increase. Hence $4 \leq g_k < A$. If $b \leq A - 1$, Lemma 6.2 applies. If $b \geq A$, no horizontal crossbar fits and the cells start V/S tiles. A V stem hits its neighbor, including an endpoint support when it points outward: all the necessary vertical segments reach its junction height, at most $a + 1 \leq b$. Only S starters remain, and adjacent ones overlap.

It remains to consider $c = 1$, so $A = a + 2$. If $b > a$, the preceding exclusions leave only H boundary contacts and a gap with $2 \leq g < A$. For $b \geq A$ the same neighbor argument applies. For $b = A - 1$, any S overlaps a neighbor; three V tiles are impossible, and when $g = 2$ their outward stems hit the two boundary stems.

Suppose $2 \leq b \leq a$. An admissible boundary V, up to reflection and translation, has the forced neighboring pair

$$\begin{aligned} V_0 :&\quad x = 0,\ 0 \leq y \leq a + 1; \quad y = 1,\ 1 \leq x \leq b, \\ H_0 :&\quad y = 0,\ 1 \leq x \leq a + 2; \quad x = a + 1,\ 1 \leq y \leq b. \end{aligned}$$

Indeed, a V with junction height a makes $(1, 1)$ impossible by the four contact types used above; height one then forces the neighboring H stem to $a+1$. For $a - b \geq 3$, the row-one gap $b+1, \ldots, a$ contradicts Lemma 6.2. For $b \geq 3$ and $a - b \in \{0, 1, 2\}$, there is a blocked run of width b in row two or three. If $a = b$, use row two immediately. If $a = b + 1$, the unique row-one gap starts V or S: it either supplies the right wall in row two, or is a low-junction leftward V whose stem fills row two, supplying the floor for row three. If $a = b + 2$, the two row-one gaps force V starters pointing outward. The left one gives these same two alternatives; the right one must have a high junction, since a low rightward stem hits H_0.

For $a \geq 7$ these arguments exclude every boundary V. Equal consecutive H junction offsets would give a forbidden gap of length $a + 1$; therefore offsets alternate between 1 and a, creating a gap of length $2a$. At most one crossbar of length $a + 2$ fits there. With none, three starters are impossible; with one, its two end runs contain $a - 2$ starters altogether, at most four. This gives $a \leq 6$, a contradiction.

The remaining fifteen cases are exactly $2 \leq b \leq a \leq 6$. Proposition 6.4 completes the proof. □

Proposition 6.4 (Computer-assisted finite boundary cases)**.** *For $2 \leq b \leq a \leq 6$, no disjoint complete copies of $P(a, b, 1, 0)$ lying in $y \geq 0$ cover the boundary patch $\{0, \ldots, 3(a+2)-1\} \times \{0, 1, 2\}$.*

Proof. The fifteen certificates allow arbitrary overhang through the three artificial sides of the patch. Their acyclic case graphs have 693–1832 nodes. Every translated, oriented tile through a queried cell is included or excluded by an explicit intersection or half-plane test; every branch ends without a possible cover. Theorem 3.2 applies. The formal entry point is `THalfPlaneFinite.finite_no_half_plane` in `HPFinite.lean`; the certificate files are `t_hpcert_a_b_1.json`, with the parameters substituted. □

Together with the positive constructions, these obstructions give half-plane tileability exactly when $b = 1$, and quadrant tileability exactly when $b = c = 1$.

7 Plane obstructions for T-polyominoes

Only $b \geq 3$ and $c \geq 2$ remain after the plane constructions. We retain $A = a + c + 1$.

Lemma 7.1 (Long-stem gaps)**.** *Assume $b \geq A$ and $c \geq 2$.*

1. *A blocked run of length $2 \leq g < A$ is impossible if both endpoint walls occupy the cells at heights a and c above the run. One such wall suffices when $g = 3$; neither extra wall is needed when $4 \leq g < A$.*
2. *A blocked run of any length $g \geq 3$ is impossible if one endpoint wall extends from row h through row $h + 1 + a$, and its supporting tile leaves the interior of row $h + 1$ unoccupied. As before, the run and its floor and endpoints satisfy the whole-tile support convention.*
3. *If $c \geq 3$, a run of length b is impossible under the weaker condition that one endpoint wall continues through row $h + 1$ and its tile leaves that row's run interior unoccupied.*

Proof. In a short run only V/S starters are possible. A V stem intersects any neighboring starter in its direction, so V can occur only at an end and point outward. A tall endpoint wall excludes that possibility. The stated lengths then force two adjacent S tiles, whose crossbars intersect.

For the second assertion, suppose a horizontal crossbar occurs and select the leftmost. It points its stem upward. Above its left arm is a run of length $r \in \{a, c\}$, with its bar as floor and its upward stem as right wall. The left wall is either the given wall or the preceding V/S starter. Such a starter has no horizontal cells one row above its bottom, since $c \geq 2$; its vertical segment reaches the required heights. Both new walls therefore satisfy the first assertion, which excludes this crossbar. Without crossbars the original run consists of starters. For $g \geq 4$ its two interior positions must be S; for $g = 3$ the deep wall excludes V at its adjacent end. Again two adjacent S tiles result. For the third assertion use the same leftmost-crossbar argument. The new short run has length $r \geq 3$; its upward stem supplies the single tall wall needed when $r = 3$, and larger r needs none. Only the additional endpoint cell in row $h+1$ is therefore required. Without crossbars, the length-b run has adjacent interior S starters since $b \geq A \geq 7$. □

Theorem 7.2. *If $a \geq c \geq 3$ and $b \geq 3$, then T does not tile the plane.*

Proof. First suppose $b \leq A - 1$. Place an upright tile O with junction $(0, 0)$. At $(1, 1)$ an existing cell immediately west and immediately south forces the covering tile to have an arm endpoint there. An upward-stem horizontal crossbar creates a blocked row of length $r \in \{a, c\}$ above its left arm. A rightward-stem vertical crossbar creates the rotated blocked run of length r. A perpendicular-stem tip creates a blocked run of length b, horizontally or vertically. All are excluded by Lemma 6.2. Thus the only possibilities are a horizontal crossbar with stem down or a vertical crossbar with stem left, with their junctions far enough away to avoid O.

The same alternatives hold at $(-1, 1)$. Both corners cannot use vertical crossbars: their columns are two units apart, and the inward stem at the lower junction intersects the other bar. Reflecting if necessary, obtain a downward-stem tile U whose crossbar is $[-A, -1] \times \{1\}$.

Now cover $p = (-1, 2)$. Its occupied lower and right neighbors again force an endpoint contact. A horizontal crossbar either hits U with its downward stem or creates a blocked row of length r with an upward stem. A horizontal perpendicular stem puts its junction above U and hits it. A vertical perpendicular stem creates a rotated blocked run of length b. A leftward-stem vertical crossbar creates a rotated run of length r. Consequently p starts a vertical crossbar V with stem right, occupying $x = -1$, $2 \le y \le A + 1$.

At $q = (-2, 2)$ the same cases exclude horizontal crossbars and leftward-stem vertical crossbars. A rightward stem or vertical perpendicular-stem cover intersects V. A horizontal perpendicular stem intersects U unless $b = A - 1$. In that last case it has junction $(-A - 1, 2)$ and fills row two from $-A$ through -2, leaving above itself a blocked run of length $A - 1$ between its crossbar and V. This is also impossible.

For $b \ge A$, use Lemma 7.1 instead. The away-pointing crossbar cases create runs of length a or c; the required deep wall is supplied by a length-b stem. The perpendicular-stem cases create runs of length b, with the same stem as floor and the adjacent bars as straight endpoint supports. The third part of the lemma applies, including its requirement one row beyond the run. Thus the antiparallel pair O, U and the forced crossbar V at p are obtained as above. Its rightward stem already hits O: its junction height is $2 + r \le a + 2 < A \le b$. □

For $c = 2$ the exceptional equality $b = a + 3$ has a plane construction. The ranges on either side require different arguments.

Proposition 7.3 (Computer-assisted short-stem obstruction)**.** *If $a \ge 2$ and $3 \le b < a + 3$, then $P(a, b, 2, 0)$ does not tile the plane.*

Proof. The symbolic certificate `t_plane_negative_corners_two_short_d10.json` has 300 nodes and 208 terminal contradictions, and adds at most seven tiles to an anchored initial tile. Its parameters satisfy only the displayed inequalities. At each node all eight orientations and an arbitrary integer junction of a tile through the selected cell are tested. The resulting exhaustive arithmetic alternatives and the geometric replay are checked in Lean. Normalize a tile in a hypothetical plane tiling to the anchor and apply Theorem 3.2. The unanchored formal theorem is `short_stem_two_arm_no_plane`. □

Theorem 7.4. *If $a \ge 2$ and $b > a + 3$, then $P(a, b, 2, 0)$ does not tile the plane.*

Proof. Here $A = a + 3$ and $b \ge A + 1$. We describe placements by their junction followed by their east, north, west, and south arms. Apply Lemma 7.1 at concave corners. An antiparallel pair, with O upright at zero and a downward-stem bar U in $[-A, -1] \times \{1\}$, is impossible: at $(-1, 2)$ horizontal bars make short deep-wall gaps, a rightward vertical bar hits O, and a leftward vertical bar makes a rotated short gap. Perpendicular-stem contacts make length-b gaps; the original stem or the bar of U supplies the deep wall.

At a concave corner of an upright tile the same tests leave only an outward vertical crossbar with lower arm two, or a vertical perpendicular-stem tip. The two concave corners cannot both use perpendicular-stem tips: their horizontal crossbars have junctions two units apart at the same height. Thus, after reflection, the corner $(1, 1)$ is covered by

$$U = (1, 3;\ b, a, 0, 2).$$

Covering $(2, 1)$ and $(2, 2)$ now forces respectively a downward-stem horizontal bar R and a leftward perpendicular stem W. Indeed, after transposition they are two starters; two S tiles overlap, inward

V stems intersect, and the upper wall excludes an outward V at the upper cell. Thus

$$R \text{ has bar } [2, A+1] \times \{1\},$$
$$W \text{ has junction } (b+2, 2) \text{ and left stem of length } b.$$

Either vertical arm order of W is allowed. Its upper-left concave corner is covered by the stem tip of U, so its lower-left corner must have the other admissible cover. This forces

$$Z = (b-1, 1;\ 2, 0, a, b).$$

Consider the b cells $[2, b+1] \times \{1\}$ below the stem of W. Their outside endpoints belong to U, W, and their two end cells belong to the distinct downward bars R, Z. If those bars overlap, we are done. Every other boundary contact is, downward, an H, V, or S. An interior V intersects its neighbor in its stem direction: a V/S neighbor has a sufficiently long vertical segment, while an H neighbor puts its stem at distance at most $a+1 < b$. An interior S must therefore have an H neighbor in the direction of its length-a arm. Avoiding overlap forces that neighbor's junction to distance $a+1$ and encloses an a-by-$(b-1)$ rectangle. No orientation fits: the two possible widths are $A > a$ and $b+1 > a$.

Hence the interval consists of consecutive downward H bars, and contains at least two since $b > A$. Consecutive junction offsets $r, s \in \{2, a\}$ enclose, immediately below their bars, a run of width $A - 1 + s - r \geq 4$. Their length-b stems provide the straight deep walls. Lemma 7.1, applied downward, gives the final contradiction. ☐

8 Obstructions for genuine crosses

Lemma 8.1. *No family of crosses with four positive arms tiles a half-plane, even if the crosses have different arm lengths.*

Proof. A boundary contact must be the tip of a downward arm. The tiles covering $(0,0)$ and $(1,0)$ have junctions $(0, r)$ and $(1, s)$, with $r, s \geq 1$. If $r \leq s$, the first tile's right arm intersects the second tile's vertical arm at $(1, r)$. If $s \leq r$, reverse the roles. Either way the tiles overlap. ☐

Theorem 8.2. *If $a, b, c, d \geq 2$, then $P(a, b, c, d)$ does not tile the plane.*

Proof. Let M be the longest arm. Normalize a tile O so its junction is zero and a longest arm points east. At each of $(\pm 1, \pm 1)$, the covering tile ends an arm. Assign the diagonal cell to the cardinal direction from which this arm arrives. Two diagonal cells cannot be assigned the same direction: for example, junctions $(r, 1), (s, -1)$ supplying both eastern cells would overlap, because the lower-reaching arm at the nearer junction meets the farther horizontal arm. All arms have length at least two, as required for this test. Thus all four directions occur exactly once.

Reflect vertically so the northeast diagonal is supplied from the east. Its covering tile has junction $(r, 1)$ with $r > M$, and west arm ending at $(1, 1)$. Since that arm has length at most M, necessarily $r = M + 1$. Its horizontal bar supplies a floor below $(1, 2)$ and $(2, 2)$; its vertical bar and that of O supply walls at $x = M + 1$ and $x = 0$. Neither cell can lie on a new horizontal bar: a junction between the walls sends an arm into the floor, while a junction outside sends its bar through a wall. Both must therefore belong to distinct vertical arms arriving from above. At the lower of their two junction heights, a horizontal arm intersects the other vertical arm. Contradiction. ☐

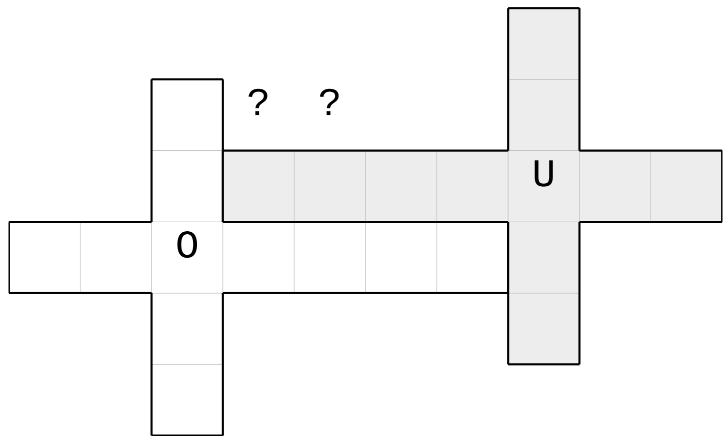

Figure 7: The trapped adjacent cells in the all-long-arm cross argument. The two supporting tiles force vertical arrivals from above, which must intersect. This local picture illustrates Theorem 8.2.

Proposition 8.3 (Computer-assisted unit-arm cross obstructions)**.** *Neither of the following families tiles the plane:*

$$P(a,1,c,d) \quad (2 \le a \le c,\ d \ge 2), \qquad P(1,1,c,d) \quad (2 \le c \le d).$$

Proof. The two symbolic trees have respectively 89 and 57 nodes. Each starts with the prototile at zero and selects $(-1,-1)$. In the adjacent-unit case there are eight first placements; in the one-unit case there are twelve. Each first placement requires one further tile, after which a selected cell has no disjoint cover. For example, an adjacent-unit branch places

$$(-1,-1-c;\ 1,c,d,1), \qquad (-2-c,-c;\ c,d,1,1),$$

and then queries $(-c, 1-c)$, which cannot be covered. All alternatives, including arbitrary integer junctions, are checked symbolically under the displayed unbounded parameter conditions. Theorem 3.2 gives the obstructions. Translation and dihedral normalization are included in the formal wrappers `CrossClassification.one_no_plane` and `CrossClassification.adjacent_no_plane`. □

Every remaining positive-arm cross has an opposite pair of unit arms. The construction for that case, together with Lemma 8.1, completes the cross classification.

9 The earlier L classification and its formal reconstruction

For $L = P(a,b,0,0)$ normalize $a \ge b \ge 1$. Its bounding dimensions are $(a+1)$ by $(b+1)$. The classification is the author's earlier result [7, 8]: $b = 1$ admits a rectangle; $b = 2$ and $(a,b) = (4,3)$ admit a strip but no quadrant; all other cases admit the plane but no half-plane. Here we summarize the negative proofs and specify where the formal reconstruction uses computer-assisted local arguments.

The quadrant paper excludes every $a \geq b \geq 2$ by local corner arguments. Although one theorem is phrased for a rectangle, its proof uses only the two adjacent boundary rays; the half-plane paper explicitly records the quadrant conclusion. The cases are separated according to equal legs, the two small $b = 2$ cases, the remaining $b = 2$ family, and $a > b \geq 3$. Endpoint contacts are excluded first; the remaining configurations force an uncovered cell or an enclosed rectangle too short for either orientation. These arguments are reproduced in the formal classification, rather than assumed as axioms.

Proposition 9.1 (Reconstructed L boundary reductions)**.** *Let $a > b \geq 3$ and $a \geq 5$. In a hypothetical half-plane tiling by L, the following consequences hold:*

1. *No boundary contact is solely a vertical tip, and no horizontal boundary leg is the short leg.*
2. *Consecutive long boundary legs have alternating stem orientations. Eight consecutive such legs can be extracted without assuming any periodicity.*
3. *Except for the separately excluded case $(a, b) = (5, 3)$, four central legs may be normalized to junctions $-1, 0, 2a + 1, 2a + 2$ in row zero. Coverage above them forces $a = b + 2$ and, up to reflection, the two row-one tiles*

$$(1, 1;\ a, b, 0, 0), \qquad (2a, 1;\ 0, a, b, 0).$$

Proof structure and computer-assisted components. The first two steps follow the boundary strategy in [7]. Once tips are excluded, normalize a proposed short boundary leg to $(0, 0; b, a, 0, 0)$. There are seven possible disjoint covers of $(1, 1)$. Six have finite obstructions. The seventh is eliminated by first applying the six established consequences to the tile covering the cell just left of the boundary leg, and then checking its remaining long-leg alternatives. This dependency order avoids circular use of the short-leg exclusion. Equal-facing consecutive long legs similarly have a finite obstruction, proving alternation.

Outside that exceptional case, the central stems bound the first-row interval $1, \ldots, 2a$. Tip and corner exclusions force its coverage by horizontal legs. Successor and endpoint counts, together with the outward-short-leg obstructions for $a = b+1$ and $a \geq b+3$, leave $a = b+2$. At that difference the endpoint count gives the displayed mixed pair or its reflection. These are symbolic reductions for unbounded arm lengths, replayed by `LHalfPlaneShortbarAssembly`, `LHalfPlaneAssembly`, and `LHalfPlaneAlternating`; their finite branches use Theorem 3.2. ☐

Proposition 9.2 (Computer-assisted final L obstructions)**.** *The mixed configuration of Proposition 9.1 is impossible for $a = b + 2$, $b \geq 4$. The case $(a, b) = (5, 3)$ also has no half-plane tiling.*

Proof. The mixed configuration uses the two specified row-one tiles and four supporting boundary tiles. Starting at $(2, 2)$, its 522-state certificate checks every possible covering placement and closes every branch. The exceptional $(5, 3)$ case has separate short-bar and alternating-eight-bar certificates with 421 and 539 states. These certificates are replayed against arbitrary whole-tile tilings; no bounded-width or periodic-tiling hypothesis is imposed.

Some leaves close with an immediate impossible cover; others use a short blocked run of length $g \leq b$. Five starters are impossible without additional caps. Four are excluded by one endpoint cap at height b; three by both such caps; two by both height-b caps and one height-a cap; even a single starter is excluded when both endpoints are capped at both heights. The supporting cells belong to already selected whole tiles, and the reflected and rotated versions check that all queries remain

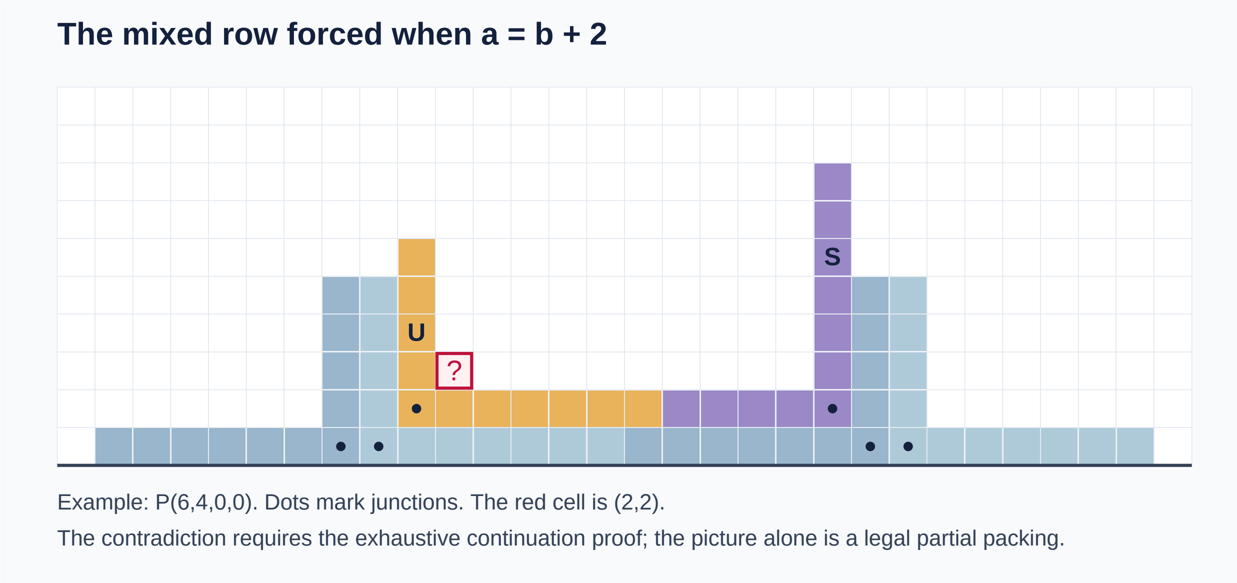

Figure 8: The mixed configuration in the final unequal-L obstruction, illustrated for $a = 6$, $b = 4$. Four boundary tiles support the mixed row; the certificate starts with the uncovered cell $(2, 2)$.

inside the half-plane. Theorem 3.2 and these local lemmas yield the claims. The formal conclusions are `LHalfPlaneMixedMotif.mixed_impossible` and `LHalfPlaneFiveThree.no_halfplane`. □

Together with the equal-leg obstruction for $a = b \geq 3$ from [7], formalized in `LHalfPlaneEqual`, the two propositions give the entire negative half-plane classification. In particular, this is a formal reconstruction of the author's prior L theorem, not a new attribution of that classification. The mixed certificate replaces the paper's final diagram argument; it is not a line-by-line formal translation of that argument.

10 Completion of the classification

Proof of Theorem 2.1. We first account for every tuple. With no positive arms, or one positive arm, the shape is a bar. Two opposite positive arms also give a bar, of length their sum plus one. Two adjacent positive arms give an L, which can be normalized to $a \geq b \geq 1$. Three positive arms give a T: rotate its missing arm south and reflect horizontally if necessary to obtain $a \geq c \geq 1$. Four positive arms give a genuine cross. Integer translations and square-grid symmetries preserve all eight capabilities, including integer-scale rep-tilings. This reduction covers degenerate tuples and does not require the parametrization to be unique.

Bars, the L shapes with $b = 1$, and the three small T shapes with $b = c = 1$, $a \leq 3$, have rectangle constructions in Section 4. Proposition 3.1 therefore gives every capability in the rectangle profile.

For the remaining L shapes, Section 4.5 gives strip tilings when $b = 2$ or $(a, b) = (4, 3)$ and plane tilings in every case. Section 9 excludes a quadrant for all $b \geq 2$, and a half-plane outside those strip cases. Thus the former have exactly the strip profile and the latter exactly the plane-only profile. In particular, absence of a quadrant excludes Rep as well as BS, HS, and R.

For a T with $b = 1$, Proposition 4.1 always gives a strip. If $c \geq 2$, Lemma 6.1 excludes a quadrant, hence gives exactly the strip profile. If $c = 1$ and $a \geq 4$, the same construction gives a bent strip, while Theorems 5.2 and 5.3 exclude a half-strip, a rectangle, and a rep-tiling. The hierarchy supplies the other four positive capabilities, giving exactly the bent-strip profile.

For a T with $b \geq 2$, Theorem 6.3 excludes a half-plane and hence every listed capability except possibly the plane. Proposition 4.2 gives a plane tiling when $c = 1$, $b = 2$, or $(c = 2,\ b = a + 3)$. In the complement, $b \geq 3$ and $c \geq 2$. When $c \geq 3$, Theorem 7.2 excludes the plane; when $c = 2$, Proposition 7.3 and Theorem 7.4 cover respectively $b < a+3$ and $b > a+3$. Thus this complement consists precisely of non-tilers.

Finally, no genuine cross tiles a half-plane, by Lemma 8.1. A cross with opposite unit arms has a plane tiling by Proposition 4.3, after rotation if necessary. If no opposite pair is unit, there are either no unit arms, exactly one, or exactly two adjacent unit arms. Theorem 8.2 excludes the first case, and Proposition 8.3, after dihedral normalization, excludes the other two. This proves both cross rows and completes every entry of the table. ☐

11 Formal verification and reproducibility

Theorem 2.1 has a complete formal proof in Lean 4.33.1 [4], using Lean's standard library. The source package contains 538 local modules and a single unconditional entry point:

```
theorem all_tuples_classified (a b c d : Nat) :
    Classified a b c d (classifyTuple a b c d)
```

This declaration belongs to the namespace `PolyominoFormal` in `MainClassification.lean`. Its only arguments are the four arm lengths. In particular, it has no unproved local-obstruction or family-classification hypothesis.

What is formalized. A lattice region is a predicate on $\mathbb{Z}^2$. A placed tile is represented by its integer junction coordinates and four directed arm lengths. `Copy` lists the eight rotations and reflections of the specified shape. `Tiling` consists of an arbitrary predicate on placed tiles, together with proofs of congruence, containment of every cell of each whole tile in the region, coverage of every region cell, and pairwise cellwise disjointness. Thus an infinite tiling need not be periodic, computable, or described by a finite motif. For finite target regions this definition also gives ordinary finite tilings: every tile contains its junction cell, and distinct tiles have distinct junction cells inside the target.

The region predicates define the full rectangle, half-strip, bent strip, quadrant, strip, half-plane, and plane. The four predicates involving widths or heights existentially quantify over all positive integer dimensions. Consequently, a half-strip impossibility theorem excludes every width, rather than a finite range examined by a search. The rep-tiling predicate quantifies over natural linear scales $k \geq 2$. Its target replaces each original cell by a $k \times k$ block; it is not obtained by multiplying the arm lengths while retaining unit thickness. The formal claim concerns congruent original tiles, integer translations, and lattice rotations and reflections, with this integer scale convention. It makes no assertion about arbitrary off-grid Euclidean dissections.

The predicate `Classified` states an equivalence for *each* of the eight capabilities. It records both positive and negative results, not merely a strongest known construction. Five distinct capability profiles occur in this family. `TupleNormalization.lean` handles rotations, reflections, bars, and zero-arm cases, including $P(0,0,0,0)$. It reduces genuine T shapes to $a \geq c \geq 1$, $b \geq 1$, and L shapes to sorted adjacent arms. The symmetry arguments transport the enlarged target as well as the small tiles when proving invariance of rep-tiling.

Mathematical result	Lean declaration and source module
Complete classification	`all_tuples_classified`, `MainClassification.lean`
Sorted L classification	`sorted_L_classified`, `MainClassification.lean`
Unequal L obstruction	`LHalfPlaneUnequal.no_halfplane`, `LHalfPlaneUnequal.lean`
Exceptional L $P(5,3,0,0)$	`LHalfPlaneFiveThree.no_halfplane`, `LHalfPlaneFiveThree.lean`
Full T classification	`normalized_T_classified`, `TClassification.lean`
$P(4,1,1,0)$, $P(5,1,1,0)$	`p4_exact_bs`, `p5_exact_bs`, `InitialTwoClassification.lean`
All $P(n,1,1,0)$	`classify_gun`, `GunFamilyClassification.lean`
Four positive arms	`classify_positive_cross`, `PositiveCrossClassification.lean`

Table 1: Principal formal statements. All declarations in this table have the namespace prefix `PolyominoFormal.`; files are relative to the archive's `lean/` directory.

Finite certificates and infinite tilings. The computer-assisted obstructions have two logically separate parts. A finite case analysis proves a local statement; a geometric argument extracts its hypotheses from an arbitrary exact tiling. Covering a chosen uncovered cell supplies an actual tile from that tiling. Completeness lemmas enumerate every possible orientation and relative placement that could cover it. Each resulting branch proves an overlap, a containment violation, another impossible configuration, or a stated successor configuration. The arithmetic obligations are proved in Lean. External search programs select useful cells and generate candidate case trees, but their answers are not assumed by the formal proof.

For $P(n,1,1,0)$, the finite corner analysis is packaged separately for $n=4$, $n=5$, and the symbolic range $n \geq 6$. These cases feed the same arbitrary-width half-strip argument. If a corner at height Y in a half-strip of height H has type s, its rank is

$$3(H-Y)+r(s), \qquad 0 \leq r(s) \leq 2.$$

Every permitted transition decreases this nonnegative integer: a transition either rises or decreases the auxiliary rank at unchanged height. This is a well-founded descent, not an inference from checking large finite rectangles. The L obstructions likewise connect finite placement arguments to actual half-plane tilings through separately proved boundary reductions. Other formal components construct infinite tilings explicitly and prove the compactness and hierarchy implications used in assembling the full profiles.

Trust and reproduction. The accepted build verified the exact source dependency closure of `MainClassification` and `GunFamilyClassification`. Lean's axiom report for `all_tuples_classified` lists exactly `propext`, `Classical.choice`, and `Quot.sound`, the standard foundational axioms used by this development. The accepted sources contain no admitted proofs, additional axioms, calls to `native_decide`, or trusted external solver answers. Certificate checking produces ordinary Lean proof terms. This assurance concerns the formal statements and their definitions; correspondence with the mathematical exposition remains a separate matter for review.

The frozen source archive is identified by an immutable repository commit. It contains the pinned toolchain, all 538 local source modules, the build driver, a file-hash manifest, build receipts, and the axiom reports. No precompiled local proof objects or external solver is needed. After installing Lean 4.33.1, extracting the archive, and entering its `polyominoes` directory, run

```
python3 src/check_lean.py MainClassification \
  GunFamilyClassification --jobs 1
```

The driver uses one worker and one Lean thread by default. It records source, dependency, compiler-version, and compiled-object hashes in an isolated output directory. An existing result is reused only when these identifiers match; specifying a new `--output` directory requests a fresh build. The receipts document the completed verification; rebuilding independently checks the proofs. The broader research archive also preserves exploratory and abandoned attempts, and is not the accepted proof dependency closure.

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The mathematical and computational role of Astra 6, operating through the Codex harness, is described in Section 1. The contributions of earlier authors to the tiling results and constructions are credited at their points of use.

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